2009 Utah State Mathematics Contest
Junior Exam Solutions

1. (e) 56.25

   $V = \pi r^2 h$. If the diameter is down 20%, the radius is down 20%. To maintain the same volume, the equation describing the adjusted lengths would be $V = \pi(\frac{4}{5} r)^2 (\frac{25}{16} h)$. Since $\frac{25}{16} = 1.5625$, there must be an increase of 56.25%.

2. (d) 3

   Let X be the initial number of coins. Since their average value is 20 cents, their total value is 20X. By adding a quarter, we get $\frac{20X + 25}{X + 1} = 21$. Solving for X, there were originally 4 coins, totaling 80 cents in value. They must have been three quarters and a nickel.

3. (e) $p^2 > -4q$

   The discriminant of the quadratic formula is $(b^2 - 4ac)$, which in this case is $(p^2 + 4q)$. Since the discriminant must be positive for a quadratic equation to have two distinct real roots, $p^2 + 4q > 0$, or $p^2 > -4q$.

4. (a) 25

   You would need to have two people born in each of the twelve months, plus one additional person, to guarantee that three people were born in the same month.

5. (d) 665

   For an integer to be both a square and a cube, it must be a perfect sixth power. $2^6 = 64$, and $3^6 = 729$. $729 - 64 = 665$ more years until her age is a square and a cube again.

6. (b) 240

   Let X be the number of voters from Salt Lake County, and Y be the number of voters from Utah County, both measured in thousands. $X + Y = 840$, and $0.65X + 0.72Y = 0.67(840)$. Solving this system of equations yields that $X = 600$, $Y = 240$.

7. (c)

3	5	4	2	1
1	3	2	5	4
4	1	5	3	2
2	4	3	1	5
5	2	1	4	3

8. (b) 1980

   If the man is alive in a year which is a square, we need a square between 1900 and 2100. The only available candidates are $44^2 = 1936$ and $45^2 = 2025$. If a man is 44 in 1936, he was born in 1892, which is in the 19th century. If a man was is 45 in 2025, he was born in 1980.

9. (a) 160

   Increasing 96 by 25% is analogous to multiplying by $\frac{5}{4}$. $96 \cdot \frac{5}{4} = 120$. Decreasing a number by 25% is analogous to multiplying by $\frac{3}{4}$; reversing this process involves multiplying by $\frac{4}{3}$. $120 \cdot \frac{4}{3} = 160$.

10. (d) 87

    $121 = 11 \cdot 11$; $51 = 3 \cdot 17$; $91 = 7 \cdot 13$; $87 = 3 \cdot 29$; $133 = 7 \cdot 19$

11. (c) 640

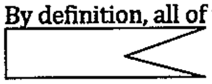
    Tide speed = X. Speed with tide = (100 + X). Speed against tide = (100 – X).
    $4(100 + X) = 16(100 - X)$. Solving for X gives that $X = 60$. Swimming 4 minutes at 160 yards per minute will move Greg 640 yards.

12. (e) 15

Of the four available additional team members, choosing each member is a yes/no choice, giving $2^4 =$ 16 different ways to select from them. However, since you must eliminate the case where you choose nobody (you must choose at least one), there are only 15 choices.

13. (c) a pentagon

By definition, all of the other shapes are convex. The following is a concave pentagon.



14. (b) 40

Cows and acres vary directly. Cows and weeks vary inversely. Acres and weeks vary directly. Let C be Cows, A be Acres, W be weeks, and K be an unknown constant. The consumption is guided by the following equation: KA = CW. Using the given data, K must be 120. If A = 15 and C = 45, then W = 40.

15. (a) $\sqrt{29}$

Using the slope, the two nearest points to (6, 2) on this line which have integer coordinates are (1, 0) and (11, 4). Using the distance formula, the distance to either point is $\sqrt{(5)^2 + (2)^2} = \sqrt{29}$.

16. (c) 17

Squaring both sides and subtracting X, $\sqrt{9 - X} = 9 - X$. Again squaring and maneuvering into standard form, $X^2 - 17X + 72 = 0$. X is either 8 or 9. Both solutions check out, and sum to 17.

17. (d) $\{x \mid 3 > x \geq 0\}$

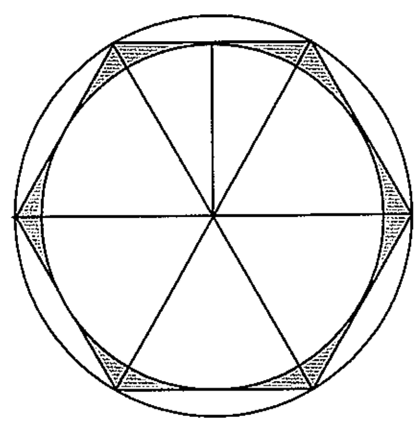
Convert into standard form and induce a common denominator to produce $\frac{x-6}{x-3} - \frac{2x-6}{x-3} \geq 0$; combining fractions, $\frac{-x}{x-3} \geq 0$, or $\frac{x}{x-3} \leq 0$. For a fraction to be less than zero, the numerator and denominator must have opposite signs. Between 0 and 3, x is positive while (x – 3) is negative. However, x cannot be 3, as this would render the fraction as undefined.

18. (e) 20

Let A = # of students in only History, B = # of students in only Algebra, C = # of students in both. (A + C) = 92, (B + C) = 85, (A + B) = 47. Solving this system of equations gives that C = 65, A = 27, and B = 20.

19. (a) $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$

A regular hexagon can be divided into six equilateral triangles. Each side length is equal to the larger radius, R. The height of each equilateral triangle is the smaller radius, equal to $\frac{\sqrt{3}}{2}$R. The shaded area is equal to the difference between the area of the hexagon and the area of the smaller circle $= 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4} R^2$. The unshaded area is equal to the area of the large circle minus the shaded area = $\pi R^2 - (6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} R^2 - \frac{3\pi}{4} R^2)$. The ratio of the two areas is, when simplified, equal to $\frac{6\sqrt{3} - 3\pi}{7\pi - 6\sqrt{3}}$.



20. (d) $7.00

Let X be the price per pound of bananas, and let Y be the price per pound of coconuts. The given information leads to 19X + 5Y = 35, 6X + 20Y = 35. Solving for X and Y, X = 1.50 and Y = 1.30. Then, 15(X + Y) = 15(2.80) = $42.00. He needs $7 more.

21. (e) 24

Let X be the distance travelled out. For uniform motion, Time = Speed/Distance, so $\frac{X}{8}$ is the time spent outbound, and $\frac{X}{3}$ is the time heading back. $\frac{X}{8} + \frac{X}{3} = 11$. Solving for X, the distance must be 24 miles.

22. (b) $4\sqrt{3}$

By the Pythagorean Theorem, BD = 13. Also by the Pythag. Thm., AD = $\sqrt{13^2 - 11^2} = \sqrt{48} = 4\sqrt{3}$

23. (a) I

$|x| \cdot |-x| = (x)\cdot(x)$ or $(-x)\cdot(-x)$, both of which = x^2.
$|-x| = (x)$ or $(-x)$, depending on the sign of x
$|x^3| - |x^2| \geq 0$, only if $|x| \geq 1$, or if $x = 0$.

24. (b) 193

Let X be the amount of time after 3 PM when the minute hand crosses the hour hand, measured in minutes. The equation to find X is: $X = 15 + \frac{X}{60} \cdot 5$. Solving, $X = \frac{180}{11}$, or $16\frac{4}{11}$, or about 16 minutes, 22 seconds. The second hand overlaps the minute hand 59 times per hour. From noon to 3 o'clock, that makes 177 overlaps. From 3PM to 3:16 PM, the second hand will overlap the minute hand 15 more times. At 3:16:22, the second hand will have passed over the minute hand one additional time (at about 3:16:16).

25. (c)

$2^{\left(2^{2^{2^2}}\right)} = 2^{2^{16}}$; $\quad (2^2)^{\left(2^{2^2}\right)} = 2^{2^5}$; $\quad$ (c) $\left(2^{2^2}\right)^{(2^2)} = 2^{2^4}$; $\quad$ (d) $\left(2^{2^{2^2}}\right)^2 = 2^{2^5}$

26. (e) 600

The fox makes 25 fox leaps in the time that the hound makes 15 hounds leaps. However, 15 hound leaps are equal in length to 27 fox leaps. So, every time the fox makes 25 fox leaps, the hound gains 2 fox leaps on the fox. Since $\frac{80}{2} = 40$, the fox will take 25 fox leaps 40 times, or make 1000 fox leaps. This will take as long as 600 hound leaps, which are equal in length to 1080 fox leaps.

27. (b) $-X - Y$

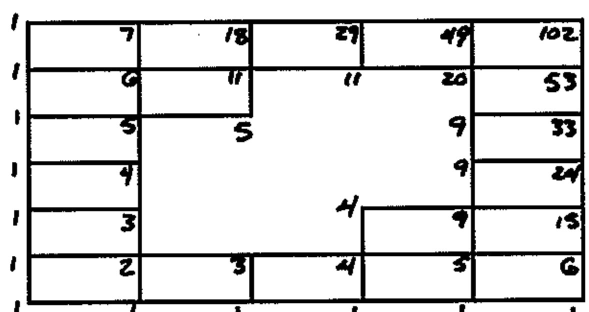
If $X < -Y$, then $X + Y < 0$. The square root of the square of a negative quantity is the opposite of the original quantity.

28. (d) 90

In numerical order, the current times are 86, 88, 94, 96, 97, and the current median is 94. If the median, after a sixth time, is to be 92, then the new value, X must be part of calculating the median. The new set of data would be 86, 88, X, 94, 96, 97. $\frac{X+94}{2} = 92$; solving for X leads to $X = 90$.

29. (c) 102

The number of paths to a specific corner is equal to the sum of the ways to arrive at the two corners which feed into that corner. The corners on the far left and bottom row each have only one path. The remaining corners have path totals working very much like Pascal's triangle.



30. (e) 91

The number of dominoes that have different numbers on the two squares would be $\frac{13 \cdot 12}{2 \cdot 1} = 78$. Then, there are thirteen more dominoes that have twin squares (squares with the same number of dominoes on each).

31. (d) $5\sqrt[10]{5}$

Using rational exponent notation and a common base of 5, you have $5^{\frac{2}{4}} \cdot 5^{\frac{3}{5}} = 5^{\frac{11}{10}} = 5\sqrt[10]{5}$

32. (b) 11

Converting 2009 to binary (base 2) notation would require knowing all of the powers of 2 less than or equal to 2009. However, for a digit count, it is only necessary to know the largest power of 2 less than 2009; $2^{10} = 1024$, while $2^{11} = 2048$. Thus, to write 2009 in binary, you would need place values from 2^0 to 2^{10}, or 11 place values/digits.

33. (a) 95

First, the foundation of the problem is that the two shortest legs of a triangle must have a sum greater than the longest leg. The pole of length 1 cannot be used at all for this reason. Let X be the length of the smallest pole in your triangle, and Y be the difference between the lengths of the middle and longest poles. For example, the triangle made with pole lengths 4, 8, 10 would have (X, Y) = (4, 2). X must be no greater than 10. The chart values indicate the number of triangles which can be created for a given (X, Y). The sum of all possible triangles is 95.

	X = 2	X = 3	X = 4	X = 5	X = 6	X = 7	X = 8	X = 9	X = 10
Y = 1	9	8	7	6	5	4	3	2	1
Y = 2		7	6	5	4	3	2	1	
Y = 3			5	4	3	2	1		
Y = 4				3	2	1			
Y = 5					1				

34. (c) $x + 4$

The Remainder Theorem states that if (x - c) evenly divides f(x), then f(c) = 0. While any of the five options are possible trial divisors, only $(-4)^3 - 2(-4)^2 - 9(-4) + 60 = 0$.

35. (e) $\frac{7}{12}$

The leather bad had probability of $\frac{3}{6}$ of trading a black marble for a red, and probability of $\frac{3}{6}$ of maintaining the same red-black mix. The probability of drawing a red now is $\frac{4}{6} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{3}{6} = \frac{21}{36} = \frac{7}{12}$.

36. (c) 4

$\sqrt{(-1) \cdot (-1)} \neq \sqrt{(-1)} \cdot \sqrt{(-1)}$. The Product Property of Radicals ($\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$) does not apply if ($\sqrt[n]{a}$) and ($\sqrt[n]{b}$) are not real numbers.

37. (b) 5

$2^{xy} = (2^x)^y = 15^y = 32$, , then (xy) must be equal to 5.

38. (a) 5π

The region bounded by the two equations is one quarter of a circle centered at the origin with radius equal to $\sqrt{20}$. The area of such a figure is $\frac{1}{4} \cdot 20\pi = 5\pi$.

39. (a) $\frac{18}{17}$

By factoring out the GCF in the numerator and denominator, you get $\frac{34!(35+1)}{34!(35-1)} = \frac{36}{34} = \frac{18}{17}$

40. (d) Green

(I) implies that if a player has a yellow card, then she also has an orange card. (II) implies that if a player has a blue card, then she also has a yellow and a red card. (IV) implies that if a player has an orange card, then she also has a blue card. (V) implies that each player must have at least one of orange, yellow, and blue. Since yellow implies orange, orange implies blue, and blue implies yellow, then every player must have yellow, orange and blue. Since each player has blue, each player also has red. Thus, all players have red, orange, yellow and blue. If one player is the only player holding all five colors, then that player uniquely holds green, as everybody else has all other colors. (III) implies only that each player has yellow and/or green.